Finite element solutions of axisymmetric Euler equations for an incompressible and inviscid fluid

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FINITE ELEMENT SOLUTIONS OF AXISYMMETRIC EULER EQUATIONS FOR AN INCOMPRESSIBLE AND INVISCID FLUID

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RESOLUTION PAR UNE METHODE D'ELEMENTS FINIS DES EQUATIONS D'UN FLUIDE INCOMPRESSIBLE NON VISQUEUX EN GEOMETRIE AXISYMMETRIQUE. APPLICATION AU CALCUL D'UNE POMPE HELICE.

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Résumé

Nous présentons dans cette étude une méthode d'élément finis pour le calcul de l'écoulement d'un fluide incompressible, non visqueux, en géométrie axisymétrique. Notre approche utilise essentiellement une formulation fonction de courant, vitesse angulaire et tourbillon des équations d'Euler et nous avons résolu les problèmes stationnaires et instationnaires. Dans le cadre d'une application au calcul d'une pompe-hélice sur l'arrière corps d'un navire, nous avons développé un modèle de calcul complet de l'interaction entre le propulseur carenné et le corps du navire. Ce modèle utilise une version simple et rapide de la méthode des caractéristiques dans un contexte éléments finis. La solution stationnaire de l'écoulement s'obtient par itérations de Picard. Les tests numériques ont mis en évidence la rapidité et la robustesse de la méthode. Des expériences réalisées au Bassin des Carènes ont révélé un très bon accord entre calcul et mesures.

FINITE ELEMENT SOLUTIONS OF AXISYMMETRIC EULER EQUATIONS FOR AN INCOMPRESSIBLE AND INVISCID FLUID

ABSTRACT

In this paper, we present a finite element method for the numerical solution of axisymmetric flows. The governing equations of the flow are the axisymmetric Euler equations. We use a stream-function angular velocity and vorticity formulation of these equations, and we consider the non stationary and the stationary problems.

For industrial applications, we have developed a general model which computes the flow past an annular airfoil and a duct propeller. It is able to take into account jumps of angular velocity and vorticity in order to model the flow in the presence of a propeller. Moreover we compute the complete flow around the after body of a ship and the interaction between a ducted propeller and the stern. In the stationnary case, we developped a simple and efficient version of the Characteristics / Finite element method. Numerical tests have shown that this last method leads to a very fast solver of the Euler equations. The numerical results are in good agreement with experimental data.

INTRODUCTION

The stream-function and vorticity formulation of the Euler equations governing an incompressible and non viscous flow has been successfully used in two dimensional problems. In a finite element context, it has been associated either with classical leap-frog or Crank-Nicolson time-differencing schemes [1,2] or with the method of characteristics [3 , 4 , 5].

It is well known that in the axisymmetric case, there is also a stream-function formulation of Euler equations. It uses the θ components, in cylindrical coordinates, of the vector potential, the velocity and the vorticity. The choice of this formulation has, in our case, many advantages. Among them we can mention the following.

First,the axisymmetric flow is completely described by three scalar functions.

Moreover the incompressibility condition is exactly satisfied.

At last, from a computational point of view, this formulation gives a simple model leading to fast solvers well adapted to our purpose: "trial and error " procedures in engineering design.

Our model involve three equations. One elliptic equation for the stream function and two transport equations for the angular velocity and the vorticity. A finite element method using non uniform meshes has been chosen in order to get a general spatial discretization giving a soft treatment of geometry. Then, the main difficulty of the numerical solution of Euler equations is to write a good solver of the transport equations. Since for applications in "pump-jet" design, we have to model the convection of jumps of angular velocity and vorticity, we need a robust method, especially well suited to difficult problems with rough conditions. This has been the key-point of this work.

One can find in [7] a general review of the numerical methods in turbomachinery flows, and in [8] and [9] some finite element applications. The present work differs from the preceding by the choice of triangular meshes, direct solutions by Choleski factorisations of the elliptic equation and an exact and direct treatment of the Kutta-Joukovski condition, elsewhere obtained through an iterative process. Our final choice of a stationary implementation of the characteristics method to solve the convection problem is the original part of this work.

This paper is organised as follows:

In sections 1 and 2 , we derive the mathematical formulations and the boundaries conditions of the problem. We precise the treatment of the Kutta-Joukovski condition.

In section 3 , we present the finite element spatial discretization and we give a convergence result in a simpler model case without "swirl".

The section 4 deals with time discretizations using Leap-Frog and Semi-Implicit Crank-Nicolson schemes. We derive theoretical stability results in both cases and we present some numerical tests showing the inability of this classical approach to model correctly the flow.

The following sections 5 and 6 are devoted to our implementation of the characteritics method giving the stationary solution of the flow by an iterative fixed-point algorithm.

At last, in section 7, we present numerical results in the case of the complete model of a duct propeller. They reveal good agreement with experiments made by B. Goirand at the Bassin des Carenes in Paris.

1 . The mathematical model

The general 3.dimensional Euler equations in cylindrical coordinates r , θ , z read :

$$\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + V_z \frac{\partial V_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad (1,a)$$

$$\frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_\theta V_r}{r} + V_z \frac{\partial V_\theta}{\partial z} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \qquad (1,b)$$

$$\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (1,c)$$

with

$$div(V) = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0 \qquad (1,d)$$

V_r, V_θ, V_z are the components of the velocity, ρ is the density of the fluid, p is the pressure.

They reduce in the axisymmetric case to the following system

$$\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + V_z \frac{\partial V_r}{\partial z} = \frac{V_\theta^2}{r} - \frac{1}{\rho}\frac{\partial p}{\partial r} \qquad (2,a)$$

$$\frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + V_z \frac{\partial V_\theta}{\partial z} = -\frac{V_\theta V_r}{r} \qquad (2,b)$$

$$\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (2,c)$$

with

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{\partial V_z}{\partial z} = 0 \qquad (2,d)$$

Remark 1 :
We did not restrict ourselves to the case $V_\theta = 0$. We just take the derivatives

$$\frac{\partial}{\partial \theta} = 0.$$

We introduce a stream-function ψ_θ such that the meridian velocity

$$V_M = (V_z, V_r)$$

can be written :

$$V_z = \frac{1}{r}\frac{\partial(r\psi_\theta)}{\partial r}, V_r = -\frac{1}{r}\frac{\partial(r\psi_\theta)}{\partial z} \tag{3}$$

Thus the zero-divergence condition (2 , d) is automatically satisfied.

Now, we consider the $\theta-$component of the vorticity vector

$$\omega_\theta = \frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} \tag{4}$$

Equations (2 , a , b , c) lead through straight forward calculations to the following system in ψ_θ , V_θ , ω_θ .

$$-\frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial(r\psi_\theta)}{\partial z}) - \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial(r\psi_\theta)}{\partial r}) = \omega_\theta \tag{5,a}$$

$$\frac{\partial(rV_\theta)}{\partial t} + V_r\frac{\partial(rV_\theta)}{\partial r} + V_z\frac{\partial(rV_\theta)}{\partial z} = 0 \tag{5,b}$$

$$\frac{\partial(\frac{\omega_\theta}{r})}{\partial t} + V_r\frac{\partial(\frac{\omega_\theta}{r})}{\partial r} + V_z\frac{\partial(\frac{\omega_\theta}{r})}{\partial z} = \frac{1}{r^2}\frac{\partial(V_\theta^2)}{\partial z} \tag{5,c}$$

With the identities (3), the above system defines completely the flow. It is the basic model of this work.

Remark 2 :

Equation (5 , a) is a simple elliptic equation in $r\psi_\theta$.

Equation (5 , b) and (5 , c) appear as transport equations of rV_θ and $\frac{\omega_\theta}{r}$, respectively, along the stream lines, with the presence of a left hand term in (5 , c).

2 . The boundary conditions

Let Ω denote in the sequel a bounded open set of R^2 with boundary Γ , such that for every point of coordinates (z , r) in Ω we have :

$$0 < r_0 \leq r \leq r_1 \tag{6}$$

The classical inviscid boundary condition

$$u.n = 0 \tag{7}$$

leads to the following condition on ψ_θ

$$curl(r\psi_\theta).n = 0 \tag{8}$$

Thus we get

$$r\psi_\theta|_{\Gamma_i} = c_i \tag{9}$$

where the c_i are constant, for each component Γ_i of the boundary Γ .

2.1 Model 1

As a theoretical model, we consider the case of a simply connected domain Ω with the boundary condition

$$r\psi_\theta|_\Gamma = 0 \tag{10}$$

2.2 Model 2

Now we turn to a more realistic case. Ω will denote the meridian section of an annular duct

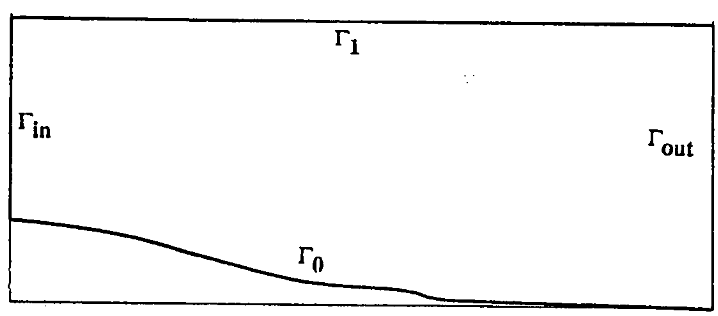

Figure 1.The axisymmetric duct

- Γ_0 and Γ_1 are supposed to be slipping walls.

- On the upstream boundary Γ_{in}, the velocity field is given.

- On the downstream boundary Γ_{out}, we only suppose that the radial component of the velocity is zero.

This model represents the flow around an axisymmetric body. The boundary Γ_1 is supposed far enough from Γ_0 to be an horizontal stream-surface. That leads to the following boundary conditions for ψ_θ , V_θ and ω_θ :

On Γ_{in} , Γ_0 and Γ_1 .

We deduce the values of ψ_θ , V_θ and ω_θ from the given velocity field. $r\psi_\theta$ being defined up to a constant, we are able to choose $r\psi_\theta = 0$ on Γ_0 . Then the law of $r\psi_\theta$ on Γ_{in} is completely known and we get the constant value c of $r\psi_\theta$ on Γ_1 .

On Γ_{out} .

The condition $V_r = 0$ leads to the homogenous Neumann boundary condition

$$\frac{\partial(r\psi_\theta)}{\partial n}|_{\Gamma_{out}} = 0 \tag{11}$$

2.3 Model 3

Let us now consider the same annular duct but with an axisymmetric airfoil shape body inside

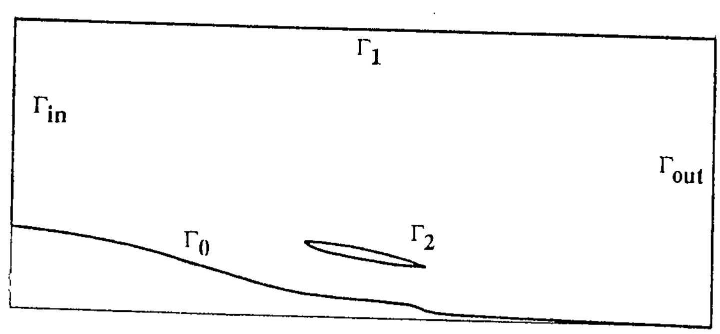

Figure 2.The complete model geometry

On the airfoil boundary Γ_2 we have the inviscid boundary condition.

$$u.n = 0 \tag{12}$$

which leads to

$$r\psi_{\theta|\Gamma_2} = c_2 \tag{13}$$

The problem is then to determine the physically correct value of the constant c_2 . This has been done by using a Kutta Joukovski condition. This condition implies the equality of the static pressures at the upper and the lower sides of the trailing edge.

We made the computation in the following manner [F. Hecht : private communication] : we looked for a stream function $r\psi_\theta$ given by

$$r\psi_\theta = \psi_0 + \alpha\psi_1 \tag{14}$$

with ψ_0 solution of equation (5 , a) at each time step, with the real boundary condition, except on Γ_2 where we take

$$\psi_{0|\Gamma_2} = 0 \tag{15}$$

and ψ_1, solution of the simple homogeneous equation

$$-\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi_1}{\partial z}\right) - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_1}{\partial r}\right) = 0 \tag{16}$$

with all the Dirichlet boundary conditions equal to zero except on Γ_2 where we take

$$\psi_{1|\Gamma_2} = 1 \tag{17}$$

The parameter α is then computed at each time step in order to satisfy the equality of the static pressures at the upper and lower sides of the trailing edge

$$P_s^+ = P_s^- \tag{18}$$

i.e.

$$|curl(\psi_0 + \alpha\psi_1)|^2 + |curl(\psi_0 + \alpha\psi_1)|^2 = \frac{2}{\rho}(P^+ - P^-) \tag{19}$$

where P^+ and P^- are the pressures on the upper and lower sides of the trailing edge.

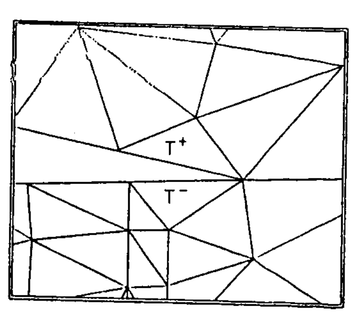

Figure 3.The numerical treatment of the Kutta - Joukovski condition.

This quadratic equation in α has two solutions. The right one is the root for which the normal velocities are opposite.

3 . Finite element approximation

In order to derive a finite element approximation of the problem, we need to introduce a variational form of the axisymmetric Euler equations.

3.1 Basic concepts and function spaces

Let (. , :) denote the axisymmetric scalar product in L^2 (Ω) .

$$(u,v) = \int\int_{\Omega} u.v.r.drdz \tag{20}$$

and

$$||\cdot||_{0,\Omega} \quad \text{the associated norm in} \quad L^2(\Omega)$$

For m $\in$ N and p $\in$ R with $1 \leq$ p $\leq + \infty$ we define the Sobolev spaces

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega); \partial^\alpha v \in L^p(\Omega), \forall|\alpha| \leq m\}$$

which is a Banach space for the norm

$$||v||_{m,p,\Omega} = (\sum_{|\alpha|\leq m} \int\int_{\Omega} |\partial^\alpha v(x)|^p rdrdz)^{\frac{1}{p}} \quad p < +\infty \tag{21}$$

or

$$||v||_{m,\infty,\Omega} = \sup_{|\alpha|\leq m} (\underset{x\in\Omega}{\text{supess}}|\partial^\alpha v(x)|) \quad p = +\infty \tag{22}$$

We also provide $W^{m,p}$ (Ω) with the following seminorm

$$|v|_{m,p,\Omega} = (\sum_{|\alpha|=m} \int_{\Omega} |\partial^\alpha v(x)|^p rdrdz)^{\frac{1}{p}} \quad \text{for} \quad p < +\infty \tag{23}$$

$$|v|_{m,\infty,\Omega} = \sup_{|\alpha|\leq m} (\underset{x\in\Omega}{\text{supess}}|\partial^\alpha v(x)|) \quad \text{for} \quad p = +\infty \tag{24}$$

In the special case p = 2 , we obtain the Hilbert spaces $H^m(\Omega)$ with the **norm** $|| \cdot ||_{m,\Omega}$ and the seminorm $| \cdot |_{m,\Omega}$.

Let us introduce the bilinear form on $(H^1(\Omega))^2$

$$a : (u, v) \rightarrow a(u, v)$$

$$a(u,v) = \int\int_{\Omega} \frac{1}{r}\left(\frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right)drdz \tag{25}$$

and the trilinear form on $W^{1,\infty}$ (Ω) x $H^1(\Omega)$ x L^2 (Ω)

$$b : (u, v, w) \rightarrow (u, v, w)$$

$$b(u,v,w) = \int\int_{\Omega} \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial r} - \frac{\partial u}{\partial r}\frac{\partial v}{\partial z}\right).w.drdz \tag{26}$$

3.2 Properties of the linear forms a and b .

3.2.1 The linear form a

Since for every point (z , r) in Ω we supposed that :

$$0 < r_0 \leq r \leq r_1$$

the bilinear form a is, as in the two dimensional case, continuous in $(H^1(\Omega))^2$ and $H_0^1(\Omega)$– Elliptic

Moreover we have the following inequalities

$$\frac{\alpha}{r_1^2}||u||_{1,r}^2 \leq \frac{1}{r_1^2}|u|_{1,\Omega}^2 \leq a(u,u) \quad \forall u \in H_0^1(\Omega) \tag{27}$$

$$|a(u,v)| \leq \frac{1}{r_0^2}|u|_{1,\Omega}|v|_{1,\Omega} \quad \forall u, v \in H^1(\Omega) \tag{28}$$

3.2.2 The form b

The trilinear form b satisfies the following properties

1) b is continuous in $W^{1,\infty}$ (Ω) x H^1 (Ω) x L^2 (Ω)

2) $\forall$ u $\in$ $W_0^{1,\infty}$ (Ω), v $\in$ H^1 (Ω) we have

$$b(u,v,v) = b(u,v,u) = 0 \tag{29}$$

Proof

$$1)|b(u,v,w)| = |\int\int_{\Omega}(\frac{\partial u}{\partial z}\frac{\partial v}{\partial r} - \frac{\partial u}{\partial r}\frac{\partial v}{\partial z}).\frac{w}{r}.rdrdz|$$

so that :

$$|b(u,v,w)| \leq \frac{1}{r_0}||\frac{\partial u}{\partial z}\frac{\partial v}{\partial r} - \frac{\partial u}{\partial r}\frac{\partial v}{\partial z}||_{0,\Omega}||w||_{0,\Omega}$$

and

$$|b(u,v,w)| \leq \frac{1}{r_0}|u|_{1,\infty,\Omega}|v|_{1,\Omega}||w||_{0,\Omega} \tag{30}$$

2) The form b is exactly the same as the form of the convective term in the two dimensional Euler equations and we refer to SAIAC [1] for the proof.

3.3 Variational formulation

Using a classical Green's formula, we obtain, in the case of model 1, the following variational formulation of equations (5 ; a , b , c)

Find a function t $\in$ [0 , T] $\rightarrow (\psi_\theta(t)\ ,\ v_\theta(t)\ ,\ \omega_\theta(t)) \in H_0^1(\Omega)$ x $H^1(\Omega)$ x $H^1(\Omega)$ such that :

$$a(r\psi_\theta(t),u) = (\frac{\omega_\theta}{r}(t),u) \quad \forall u \in H_0^1(\Omega) \tag{31,a}$$

$$E \qquad (\frac{d}{dt}rv_\theta(t),v) = b(r\psi_\theta(t),rv_\theta(t),v) \quad \forall v \in H^1(\Omega) \tag{31,b}$$

$$(\frac{d}{dt}\frac{\omega_\theta}{r}(t),w) = b(r\psi_\theta(t),\frac{\omega_\theta}{r}(t),w) + (\frac{1}{r^2}\frac{\partial}{\partial z}(v_\theta^2(t)),w) \quad \forall w \in H^1(\Omega) \tag{31,c}$$

3.4 Conservations properties of problem E

Using the fondamental property (29) of the trilinear form b we derive the following results :

$$1) \qquad \frac{1}{2}\frac{d}{dt}||rv_\theta(t)||_{0,\Omega}^2 = b(r\psi_\theta(t),rv_\theta(t),rv_\theta(t)) = 0 \qquad \forall t \in [0,T] \tag{32}$$

Thus we get the conservation of the L^2 norm of $rv_\theta(t)$

$$||rv_\theta(t)||_{0,\Omega} = ||rv_\theta(0)||_{0,\omega} \qquad \forall t \in [0,T] \tag{33}$$

$$2) \qquad \frac{1}{2}\frac{d}{dt}||\frac{\omega_\theta}{r}(t)||_{0,\Omega}^2 = b(r\psi_\theta(t),\frac{\omega_\theta}{r}(t),\frac{\omega_\theta}{r}(t)) + (\frac{1}{r^2}\frac{\partial}{\partial z}(v_\theta^2(t)),\frac{\omega_\theta}{r}(t)) \tag{34}$$

so that :

$$\frac{d}{dt}||\frac{\omega_\theta}{r}(t)||_{0,\Omega} \leq \frac{1}{r_0^2}||\frac{\partial}{\partial z}(v_\theta^2(t))||_{0,\Omega} \tag{35}$$

and finally

$$||\frac{\omega_\theta}{r}(t)||_{0,\Omega} \leq ||\frac{\omega_\theta}{r}(0)||_{0,\Omega} + \frac{1}{r_0^2}\int_0^t ||\frac{\partial}{\partial z}(v_\theta^2(t))||_{0,\Omega} dt \tag{36}$$

Remark : In the particular case of an initial value of v_θ equal to zero, v_θ remains null for all t $\in$ [0 , T] and the system E reduces to two coupled equations involving $r\psi_\theta$ and $\frac{\omega_\theta}{r}$.

Moreover in that case, we get

$$||\frac{\omega_\theta}{r}(t)||_{0,\Omega} = ||\frac{\omega_\theta}{r}(0)||_{0,\Omega} \qquad \forall t \in [0, T]. \tag{37}$$

3.5 Generalization

3.5.1 In the case of model 2 the test function space in the first elliptic equation (31 , a) is replaced by the space V defined by :

$$V = \{v \in H^1(\Omega); v = 0 \quad \text{on} \quad \Gamma_{in} \cup \Gamma_0 \cup \Gamma_1\}$$

The unknown stream-function $r\psi_\theta$ satisfies the following boundary conditions

$r\psi_\theta$ given on Γ_{in}

$r\psi_\theta = 0$ on Γ_0 and $r\psi_\theta$ = c (given constant) on Γ_1

$\frac{\partial(r\psi_\theta)}{\partial n} = 0$ on Γ_{out}

3.5.2 In the case of model 3, we moreover have to take into account a Kutta Joukovski condition (see paragraph 2.3). We made the computations as follows :

The stream function ψ_1 has been computed once and for all at the beginning of the program.
It does not depend on time. Then we just have to solve one elliptic equation at each time step to get the complete stream-function.

3.6 Finite element spatial approximation

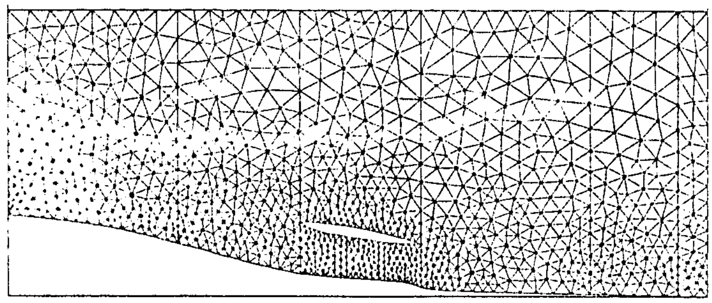

Figure 4. The computational mesh in the case of the complete model.

Let U_h and V_h be two finite dimensional spaces such that $U_h \subset W_0^{1,\infty}(\Omega)$ and $V_h \subset W^{1,\infty}(\Omega)$.

We approximate the continuous problem (E) by the following approximate problem (E_h) .

Find a function t $\in$ [0 , T] $\rightarrow$ ($\psi_{\theta,h}(t)$, $v_{\theta,h}(t)$, $\omega_{\theta,h}(t)$) $\in U_h \times V_h \times V_h$ satisfying for all t $\in$ [0,T]

$$E_h \quad \begin{cases} a(r\psi_{\theta,h}(t), u_h) = (\frac{\omega_{\theta,h}}{r}(t), u_h) \qquad \forall u_h \in U_h & (38, a) \\ (\frac{d}{dt} r v_{\theta,h}(t), v_h) = b(r\psi_{\theta,h}(t), r v_{\theta,h}(t), v_h) \qquad \forall v_h \in V_h & (38, b) \\ (\frac{d}{dt}\frac{\omega_{\theta,h}}{r}(t), w_h) = b(r\psi_{\theta,h}(t), \frac{\omega_{\theta,h}}{r}(t), w_h) + (\frac{1}{r^2}\frac{\partial}{\partial z}(v_{\theta,h}^2(t)), w_h) \qquad \forall w_h \in V_h & (38, c) \end{cases}$$

3.6.1 Conservation properties of problem E_h

We get for the approximate solutions $r\psi_{\theta,h}$, $rv_{\theta,h}$ and $\frac{w_{\theta,h}}{r}$ the same bounds as for the exact solution.

For instance :

$$||rv_{\theta,h}(t)||_{0,\Omega} = ||rv_{\theta,h}(0)||_{0,\Omega} \qquad \forall t \in [0, T]$$

3.6.2 A first convergence result

Let us consider the simplest model problem E^* in the particular case of $V_\theta = 0$, (flow without "swirl").

The corresponding approximate problem E_h^* reduces to the following system of two equations in ψ_θ and ω_θ .

$$E_h^* \quad \begin{cases} a(r\psi_{\theta,h}(t), u_h) = (\frac{\omega_{\theta,h}}{r}(t), u_h) \qquad \forall u_h \in U_h & (39,a) \\ (\frac{d}{dt}\frac{\omega_{\theta,h}}{r}(t), v_h) = b(r\psi_{\theta,h}(t), \frac{\omega_{\theta,h}}{r}(t), v_h) \qquad \forall v_h \in V_h & (39,b) \end{cases}$$

Assume that $(\psi_\theta, \omega_\theta)$ solution of the problem E^* belongs to the space

$$L^\infty(0,T;[W^{k+1,\infty}(\Omega) \cup W_0^{1,\infty}(\Omega)] \times W^{k+1,\infty}(\Omega))$$

Then under classical hypotheses of finite element interpolation we get the following error bound

$$|\psi_\theta(t) - \psi_{\theta,h}(t)|_{1,\Omega} + ||\omega_\theta(t) - \omega_{\theta,h}(t)||_{0,\Omega} \le Ch^k \qquad (40)$$

Proof :

The proof follows the same lines as the proof of the convergence of the finite element method in the two dimensional case, see SAIAC [1]. In fact the form b here is exactly the same as in the two dimensional case and there are *just* slight differences in the expressions of the bilinear form a and the scalar product.

These differences are very easy to handle since we can assume that hypotheses (6) hold.

Remark : The general case.

The general case is more tricky because of the term

$$(\frac{1}{r^2}\frac{\partial}{\partial z}(v_{\theta,h}^2(t)), w_h)$$

in the equation (38 , c).

We did not succeed already to proove the convergence of the finite element scheme in that case.

4. Time discretization

Let us choose a positive integer N , let Δt denote the corresponding time-step

$$\Delta t = \frac{T}{N}$$

and (t_n) the subdivision of [0 , T]

$$t_n = n.\Delta t \qquad \text{for} \quad 0 \leq n \leq N$$

let $\psi^n_{\theta,h}$, $v^n_{\theta,h}$ and $\omega^n_{\theta,h}$ denote approximations of $\psi_\theta(t_n)$, $v_\theta(t_n)$ and $\omega_\theta(t_n)$ respectively.

4.1 The Leap-frog scheme

The leap-frog scheme for the problem E can be written as follows

$$a(r\psi^n_{\theta,h}, u_h) = (\frac{\omega^n_{\theta,h}}{r}, u_h) \qquad \forall u_h \in U_h \tag{41,a}$$
$$(rv^{n+1}_{\theta,h} - rv^{n-1}_{\theta,h}, v_h) = 2\Delta t \quad b(r\psi^n_{\theta,h}, rv^n_{\theta,h}, v_h) \qquad \forall v_h \in V_h \tag{41,b}$$
$$(\frac{\omega^{n+1}_{\theta,h}}{r} - \frac{\omega^{n-1}_{\theta,h}}{r}, w_h) = 2\Delta t \quad [b(r\psi^n_{\theta,h}, \frac{\omega^n_{\theta,h}}{r}, w_h) + (\frac{1}{r^2}\frac{\partial}{\partial z}(v^n_{\theta,h})^2, w_h)] \qquad \forall w_h \in V_h \tag{41,c}$$

with $\psi^0_{\theta,h}$, $v^0_{\theta,h}$ and $\omega^0_{\theta,h}$ given
and $\psi^1_{\theta,h}$, $v^1_{\theta,h}$ and $\omega^1_{\theta,h}$ solutions of

$$a(r\psi^1_{\theta,h}, u_h) = (\frac{\omega^1_{\theta,h}}{r}, u_h) \qquad \forall u_h \in U_h \tag{41,d}$$
$$(rv^1_{\theta,h} - rv^0_{\theta,h}, v_h) = \Delta t \quad b(r\psi^0_{\theta,h}, rv^0_{\theta,h}, v_h) \qquad \forall v_h \in V_h \tag{41,e}$$
$$(\frac{\omega^1_{\theta,h}}{r} - \frac{\omega^0_{\theta,h}}{r}, w_h) = \Delta t \quad [b(r\psi^0_{\theta,h}, \frac{\omega^0_{\theta,h}}{r}, w_h) + (\frac{1}{r^2}\frac{\partial}{\partial z}(v^0_{\theta,h})^2, w_h)] \qquad \forall w_h \in V_h \tag{41,f}$$

The stability of the Leap-Frog scheme follows from the next lemma.

4.1.1 Lemma

Under the following stability hypotheses :

$$1) \qquad c\Delta t[\frac{1}{h}|r\psi^n_{\theta,h}|_{1,\infty,\Omega} + \frac{1}{r^4}|rv^n_{\theta,h}|_{1,\infty,\Omega}] < 1 \qquad \forall n \in 0, N \tag{42}$$

2) there exists a constant A > 0 such that

$$\frac{C}{h}|r\psi_{\theta,h}^{n} - r\psi_{\theta,h}^{n-1}|_{1,\infty,\Omega} + \frac{c}{r^4}(|rv_{\theta,h}^{n}|_{1,\infty,\Omega} + |rv_{\theta,h}^{n+1}|_{1,\infty,\Omega}) < A. \tag{43}$$

We have the following bound for every n = 0 , .. N

$$||rv_{\theta,h}^{n}||_{0,\Omega}^2 + ||\frac{\omega_{\theta,h}^{n}}{r}||_{0,\Omega}^2 \leq C(||rv_{\theta,h}^{0}||_{0,\Omega}^2 + ||rv_{\theta,h}^{1}||_{0,\Omega}^2 + ||\frac{\omega_{\theta,h}^{0}}{r}||_{0,\Omega}^2 + ||\frac{\omega_{\theta,h}^{1}}{r}||_{0,\Omega}^2) \tag{44}$$

Proof :

Let us introduce

$$S_n = ||rv_{\theta,h}^{n}||_{0,\Omega}^2 + ||rv_{\theta,h}^{n+1}||_{0,\Omega}^2 + ||\frac{w_{\theta,h}^{n}}{r}||_{0,\Omega}^2 + ||\frac{w_{\theta,h}^{n+1}}{r}||_{0,\Omega}^2$$

$$-2\Delta t.b(r\psi_{\theta,h}^{n}, rv_{\theta,h}^{n}, rv_{\theta,h}^{n+1}) - 2\Delta t.b(r\psi_{\theta,h}^{n}, \frac{\omega_{\theta,h}^{n}}{r}, \frac{\omega_{\theta,h}^{n+1}}{r}) - 2\Delta t(\frac{1}{r^2}\frac{\partial}{\partial z}(v_{\theta,h}^{n})^2, \frac{\omega_{\theta,h}^{n+1}}{r})$$

We follow, as in ref [ ] the energy method used by Richtmyer and Morton.

We have :

$$|b(r\psi_{\theta,h}^{n}, rv_{\theta,h}^{n}, rv_{\theta,h}^{n+1})| \leq C|r\psi_{\theta,h}^{n}|_{1,\infty,\Omega}|rv_{\theta,h}^{n}|_{1,\Omega}||rv_{\theta,h}^{n+1}||_{0,\Omega}$$

and by use of the inverse inequality :

$$|b(r\psi_{\theta,h}^{n}, rv_{\theta,h}^{n}, rv_{\theta,h}^{n+1})| \leq \frac{C}{h}|r\psi_{\theta,h}^{n}|_{1,\infty,\Omega}||rv_{\theta,h}^{n}||_{0,\Omega}||rv_{\theta,h}^{n+1}||_{0,\Omega}$$

Similarly, we get :

$$|b(r\psi_{\theta,h}^{n}, \frac{\omega_{\theta,h}^{n}}{r}, \frac{\omega_{\theta,h}^{n+1}}{r})| \leq \frac{C}{h}|r\psi_{\theta,h}^{n}|_{1,\infty,\Omega}||\frac{\omega_{\theta,h}^{n}}{r}||_{0,\Omega}||\frac{\omega_{\theta,h}^{n+1}}{r}||_{0,\Omega}$$

and

$$|(\frac{1}{r^2}\frac{\partial}{\partial z}(v_{\theta,h}^{n})^2, \frac{\omega_{\theta,h}^{n+1}}{r})| \leq \frac{c}{r_0^4}|rv_{\theta,h}^{n}|_{1,\infty,\Omega}||rv_{\theta,h}^{n}||_{0,\Omega}||\frac{\omega_{\theta,h}^{n+1}}{r}||_{0,\Omega}$$

so that

$$(1 - C\frac{\Delta t}{h}|r\psi^n_{\theta,h}|_{1,\infty,\Omega} - c\frac{\Delta t}{r^4}|rv^n_{\theta,h}|_{1,\infty,\Omega}).(||rv^n_{\theta,h}||^2_{0,\Omega} + ||rv^{n+1}_{\theta,h}||^2_{0,\Omega} + ||\frac{\omega^n_{\theta,h}}{r}||^2_{0,\Omega} + ||\frac{\omega^{n+1}_{\theta,h}}{r}||^2_{0,\Omega})$$

$$\leq S_n \leq (1 + C\frac{\Delta t}{h}|r\psi^n_{\theta,h}|_{1,\infty,\Omega} + c\frac{\Delta t}{r^4}|rv^n_{\theta,h}|_{1,\infty,\Omega})(||rv^n_{\theta,h}||^2_{0,\Omega} + ||rv^{n+1}_{\theta,h}||^2_{0,\Omega} + ||\frac{\omega^n_{\theta,h}}{r}||^2_{0,\Omega} + ||\frac{\omega^{n+1}_{\theta,h}}{r}||^2_{0,\Omega}).$$

But we also have

$$S_n - S_{n-1} = ||rv^{n+1}_{\theta,h}||^2_{0,\Omega} - ||rv^{n-1}_{\theta,h}||^2_{0,\Omega} + ||\frac{\omega^{n+1}_{\theta,h}}{r}||^2_{0,\Omega} - ||\frac{\omega^{n-1}_{\theta,h}}{r}||^2_{0,\Omega}$$

$$-2\Delta t.b(r\psi^n_{\theta,h}, rv^n_{\theta,h}, rv^{n+1}_{\theta,h}) + 2\Delta t b(r\psi^{n-1}_{\theta,h}, rv^{n-1}_{\theta,h}, rv^n_{\theta,h})$$

$$-2\Delta t.b(r\psi^n_{\theta,h}, \frac{\omega^n_{\theta,h}}{r}, \frac{\omega^{n+1}_{\theta,h}}{r}) + 2\Delta t.b(r\psi^{n-1}_{\theta,h}, \frac{\omega^{n-1}_{\theta,h}}{r}, \frac{\omega^n_{\theta,h}}{r})$$

$$-2\Delta t.(\frac{1}{r^2}\frac{\partial}{\partial z}(v^n_{\theta,h})^2, \frac{\omega^{n+1}_{\theta,h}}{r}) + 2\Delta t.(\frac{1}{r^2}\frac{\partial}{\partial z}(v^{n-1}_{\theta,h})^2, \frac{\omega^n_{\theta,h}}{r})$$

using then :

$$||rv^{n+1}_{\theta,h}||^2_{0,\Omega} - ||rv^{n-1}_{\theta,h}||^2_{0,\Omega} = 2\Delta t.b(r\psi^n_{\theta,h}, rv^n_{\theta,h}, rv^{n-1}_{\theta,h} + rv^{n+1}_{\theta,h})$$

and

$$||\frac{\omega^{n+1}_{\theta,h}}{r}||^2_{0,\Omega} - ||\frac{\omega^{n-1}_{\theta,h}}{r}||^2_{0,\Omega} = 2\Delta t.[b(r\psi^n_{\theta,h}, \frac{\omega^n_{\theta,h}}{r}, \frac{\omega^{n-1}_{\theta,h}}{r} \frac{+\omega^{n+1}_{\theta,h}}{r}) + (\frac{1}{r^2}\frac{\partial}{\partial z}(v^n_{\theta,h})^2, \frac{\omega^{n-1}_{\theta,h}}{r} + \frac{\omega^{n+1}_{\theta,h}}{r})]$$

we get

$$S_n - S_{n-1} = 2\Delta t \quad [b(r\psi^n_{\theta,h} - r\psi^{n-1}_{\theta,h}, rv^n_{\theta,h}, rv^{n-1}_{\theta,h}) + b(r\psi^n_{\theta,h} - r\psi^{n-1}_{\theta,h}, \frac{\omega^n_{\theta,h}}{r}, \frac{\omega^{n-1}_{\theta,h}}{r})$$

$$+(\frac{1}{r^2}\frac{\partial}{\partial z}(v^n_{\theta,h})^2, \frac{\omega^{n-1}_{\theta,h}}{r}) + (\frac{1}{r^2}\frac{\partial}{\partial z}(v^{n-1}_{\theta,h})^2, \frac{\omega^n_{\theta,h}}{r})]$$

So that we get

$$S_n - S_{n-1} \le \frac{2c\Delta t}{h} |r\psi_{\theta,h}^n - r\psi_{\theta,h}^{n-1}|_{1,\infty,\Omega} \quad [||rv_{\theta,h}^n||_{0,\Omega}||rv_{\theta,h}^{n-1}||_{0,\Omega} + ||\frac{\omega_{\theta,h}^n}{r}||_{0,\Omega}]||\frac{\omega_{\theta,h}^{n-1}}{r}||_{0,\Omega}]$$

$$+\frac{2c\Delta t}{r_0^4}[|rv_{\theta,h}^n|_{1,\infty,\Omega}||rv_{\theta,h}^n||_{0,\Omega}||\frac{\omega_{\theta,h}^{n-1}}{r}||_{0,\Omega} + |rv_{\theta,h}^{n-1}|_{1,\infty,\Omega}||rv_{\theta,h}^{n-1}||_{0,\Omega}||\frac{\omega_{\theta,h}^n}{r}||_{0,\Omega}]$$

$$S_n - S_{n-1} \le [\frac{c\Delta t}{h}|r\psi_{\theta,h}^n - r\psi_{\theta,h}^{n-1}|_{1,\infty,\Omega} + \frac{c\Delta t}{r^4}(|rv_{\theta,h}^n|_{1,\infty,\Omega} + |rv_{\theta,h}^{n-1}|_{1,\infty,\Omega})].$$

$$[||rv_{\theta,h}^n||_{0,\Omega}^2 + ||rv_{\theta,h}^{n-1}||_{0,\Omega}^2 + ||\frac{\omega_{\theta,h}^n}{r}||_{0,\Omega}^2 + ||\frac{\omega_{\theta,h}^{n-1}}{r}||_{0,\Omega}^2]$$

and by use of the stability hypotheses we derive

$$||rv_{\theta,h}^n||_{0,\Omega}^2 + ||rv_{\theta,h}^{n-1}||_{0,\Omega}^2 + ||\frac{\omega_{\theta,h}^n}{r}||_{0,\Omega}^2 + ||\frac{\omega_{\theta,h}^{n-1}}{r}||_{0,\Omega}^2 \le KS_0$$

$$+AK\Delta t \sum_{m=1}^{n} [||rv_{\theta,h}^m||_{0,\Omega}^2 + ||rv_{\theta,h}^{m-1}||_{0,\Omega}^2 + ||\frac{\omega_{\theta,h}^m}{r}||_{0,\Omega}^2 + ||\frac{\omega_{\theta,h}^{m-1}}{r}||_{0,\Omega}^2]$$

and the result

4.2 A Semi implicit scheme of order two

This scheme is a semi-implicit CRANK - NICOLSON scheme. It can be written as follow in a finite element context :

$$a(r\psi_{\theta,h}^n, u_h) = (\frac{\omega_{\theta,h}^n}{r}, u_h) \quad \forall u_h \in U_h \tag{45,a}$$

$$a(r\psi_{\theta,h}^{\frac{n+1}{2}}, u_h) = (\frac{\omega_{\theta,h}^n}{r}, u_h) + \frac{\Delta t}{2}[b(r\psi_{\theta,h}^n, \frac{\omega_{\theta,h}^n}{r}, u_h) + (\frac{1}{r^2}\frac{\partial}{\partial z}(v_{\theta,h}^n)^2, u_h)] \quad \forall u_h \in U_h \tag{45,b}$$

$$(rv_{\theta,h}^{n+1} - rv_{\theta,h}^n, v_h) = \frac{\Delta t}{2} b(r\psi_{\theta,h}^{\frac{n+1}{2}}, rv_{\theta,h}^n + rv_{\theta,h}^{n+1}, v_h) \quad \forall v_h \in V_h \tag{45,c}$$

$$(\frac{\omega_{\theta,h}^{n+1}}{r} - \frac{\omega_{\theta,h}^n}{r}, w_h) = \frac{\Delta t}{2}[b(r\psi_{\theta,h}^{\frac{n+1}{2}}, \frac{\omega_{\theta,h}^n}{r} + \frac{\omega_{\theta,h}^{n+1}}{r}, w_h) + (\frac{1}{r^2}(\frac{\partial}{\partial z}(v_{\theta,h}^n)^2 + \frac{\partial}{\partial z}(v_{\theta,h}^{n+1})^2), w_h)]\forall w_h \in V_h \tag{45,d}$$

This scheme is of order two in time and it satisfies the following stability property

Let us assume that it exists a constant A such that we get the inequality :

$$\left\|\frac{1}{r^2}\frac{\partial}{\partial z}(v^n_{\theta,h})^2+\frac{1}{r^2}\frac{\partial}{\partial z}(v^{n+1}_{\theta,h})^2\right\|_{0,\Omega}\leq A \tag{46}$$

the preceding scheme is stable and we easily get :

$$\|rv^n_{\theta,h}\|_{0,\Omega}=\|rv^0_{\theta,h}\|_{0,\Omega}\quad \forall n=0,1,...N \tag{47}$$

$$\left\|\frac{\omega^n_{\theta,h}}{r}\right\|_{0,\Omega}\leq\left\|\frac{\omega^0_{\theta,h}}{r}\right\|_{0,\Omega}+\frac{AT}{2} \tag{48}$$

4.3 Methods of characteristics

For the application of the method of characteristics to the transport equations (5,b) and (5,c) in a finite element context we refer to El Dabaghi and Saiac [6]

4.4 Numerical tests

The numerical tests have shown that the explicit Leap-frog scheme requires very small time steps in order to satisfy the stability condition. The semi-implicit scheme is better. Although it is more costly for each time step, it can work with much larger Δt and globally, it is faster.

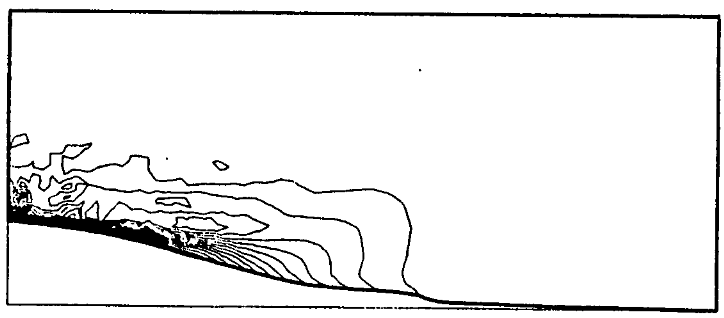

Figure 5.The vorticity ($\frac{\omega_\theta}{r}$) field at t=1 computed by the Leap-frog scheme with $\Delta t = 0.001$

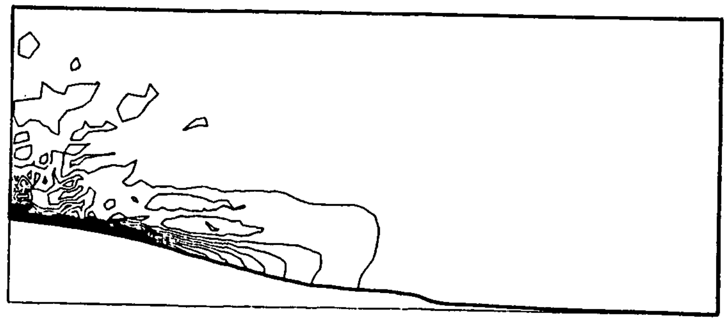

Figure 6.The vorticity ($\frac{\omega_\theta}{r}$) field at t=1 computed by the semi-implicit scheme with $\Delta t = 0.01$

However the best results for the time dependent transport equation were obtained by the use of the characteristics method.See a comparison of this schemes in ref [6]

5. An iterative method for the stationary solution

In order to get the stationary solution of EULER equations (5.a, b, c) we can of course use the time discretization schemes quoted above. But it is a better choice with respect to numerical stability and computational time to use the following simple version of the characteristics method. This implementation essentially uses our easy knowledge of the stream lines, which is the case for plane or axisymmetric flow problems.

Let us consider the simplest model without swirl (i.e. with $V_\theta = 0$) to explain the method. In that case the problem reduces to the problem E* , the vortex $\frac{\omega_\theta}{r}$ is simply convected along the stream lines $r\psi_\theta =$ cste.

Thus to detemine $\frac{\omega_\theta}{r}$ at any point x of the domain Ω we just have to find on which stream lines lies the point x . Then we go back along this stream-line to the entry point on the upstream boundary where the value of the vortex is given.

We can summarize the computational process by the following iteration method.

Suppose that $\psi^0_{\theta,h}$ is given at the time t_0. Then for any n ≥ 0 , define $\psi^{n+1}_{\theta,h}$ from $\psi^n_{\theta,h}$ by :

$$a(r\psi^{n+1}_{\theta,h}, u_h) = (\frac{\omega^n_{\theta,h}}{r}, u_h) = (\omega_i(\psi^n_{\theta,h}), u_h) \quad \forall u_h \in U_h \tag{49}$$

where ω_i is the numerical function, defined from the given upstream boundary values of the flow, which gives the functional law between the values of $r\psi_\theta$ and those of $\frac{\omega_\theta}{r}$.

More generally, we can consider the family of algorthms

$\psi^0_{\theta,h}$ given at t_0

Then $\psi^{n+1}_{\theta,h}$ is computed from $\psi^n_{\theta,h}$ by :

$$a(r\psi^{n+1}_{\theta,h}, u_h) = a(r\psi^n_{\theta,h}, u_h) - \rho(a(r\psi^n_{\theta,h}, u_h) - (\omega_i(r\psi^n_{\theta,h}), u_h)) \quad \forall u_h \in U_h \tag{50}$$

If $\rho = 1$ we recover (49)

Moreover, if ω_i is differentiable, we should be able to solve the problem by a Newton's method such as the following :

$\psi^0_{\theta,h}$ given at t_0

$$a(r\psi^{n+1}_{\theta,h}, u_h) - (\omega_i'(r\psi^n_{\theta,h}).r\psi^{n+1}_{\theta,h}, u_h) = (\omega_i(r\psi^n_{\theta,h}), u_h) - (\omega_i'(r\psi^n_{\theta,h}).r\psi^n_{\theta,h}, u_h) \quad \forall u_h \in U_h \tag{51}$$

This kind of iterations has been studied by many authors. One's can find an interesting discussion in Glowinski [10]. See also Eydeland and Turkington [11]

First of all we have the following theoretical result. See Brezis, Sibony [12]

Let us consider the following non linear problem.

Find u $\in$ U such that

$$a(u, v) = (\omega(u), v) \quad \forall v \in U \tag{52}$$

Where a is a bilinear, continuous and strongly elliptic form which satisfies :

$$\alpha ||u||^2 \leq a(u, u) \quad \forall u \in U \quad \text{with} \quad \alpha > 0 \tag{53}$$

$$|a(u, v)| \leq M ||u|| |v|| \quad \forall u \quad \text{and} \quad v \in U \tag{54}$$

and ω is a non linear operator in U

Let us define A : U $\rightarrow$ U by

$$(A(u), v) = a(u, v) - (\omega(u), v) \quad \forall u \quad \text{and} \quad v \in U \tag{55}$$

We have the following result

Theorem :
Suppose A is Lipschitz continuous on the bounded sets of U and suppose that A is strongly elliptic, i.e there exists a constant k > 0 such that

$$(A(u_2) - A(u_1), u_2 - u_1) \geq k ||u_2 - u_1||^2 \qquad \forall u_1 \quad \text{and} \quad u_2 \in U \tag{56}$$

Then the problem (52) has a unique solution. Moreover the following iteration

u^0 given

u^{n+1} defined from u^n by

$$a(u^{n+1}, v) = a(u^n, v) - \rho(a(u^n, v) - (\omega(u^n), v)) \qquad \forall v \in U \tag{57}$$

converge to the solution u of (52) for every constant ρ satisfying

$$0 < \rho < \rho_M \tag{58}$$

ρ_M being a positive constant depending on u^0 in general

Let us make some comments on the ellipticity condition (56).

In our case, it implies there exists a positive number k such that, for all u_1 and u_2 in U we have :

$$a(u_2 - u_1, u_2 - u_1) - (\omega_i(u_2) - \omega_i(u_1), u_2 - u_1) \geq k||u_2 - u_1||^2 \tag{59}$$

But a is strongly elliptic, with :

$$a(u_2 - u_1, u_2 - u_1) \geq \alpha||u_2 - u_1||^2 \tag{60}$$

Then the inequality will be obtained if we can ensure that

$$(\omega_i(u_2) - \omega_i(u_1), u_2 - u_1) \leq 0 \qquad \forall u_1 \quad \text{and} \quad u_2 \in U \tag{61}$$

or if we can suppose that ω_i satisfies a Lipschitz condition

$$||\omega_i(u_2) - \omega_i(u_1)|| \leq L||u_2 - u_1|| \tag{62}$$

with a Lipschitz constant L such that $L < \alpha$

Let us remark that in this last case, it is easy to proove the convergence of the iterations (49) by a contraction argument .

We don't know whether the condition (59) is a necessary condition for the existence or the stability of the flow. And we refer to Arnold [13] for further considerations on stability of stationary solutions of Euler equations . Let us however point out the following problem :

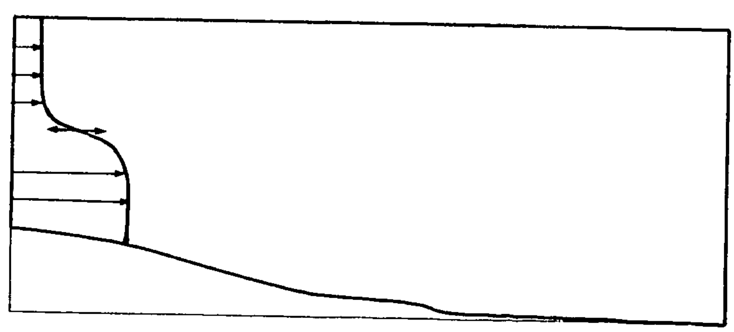

Figure 7.

In the case of a flow with a given velocity profile such that ω_i' is not bounded, we do have instability of the flow . And this is one of the main difficulty of the modelization of a propeller .

6 . Fixed - point iteration algorithm versus time - dependent approach .

Let us denote by $\psi_{\theta,h}$ the stationary solution of the approximate problem E $_h$. When the fixed-point algorithm is convergent, we get the following error bound for every iteration n = 0,...N

$$||\psi^n_{\theta,h} - \psi_{\theta,h}||_{1,\Omega} \leq Ck^n||\psi^0_{\theta,h} - \psi_{\theta,h}|| \tag{63}$$

with k $\leq$1

The convergence rate depends on the value of k, but we shall get for sufficiently large N

$$||\psi^n_{\theta,h} - \psi_{\theta,h}||_{1,\Omega} \leq \epsilon \tag{64}$$

whatever ϵ be . Then the global error is only a finite element interpolation error.

On the contrary, in a time dependent approach, each iteration corresponds to a time step . We solve a differential equation of the following kind .

$$\frac{d}{dt}\psi_{\theta,h} = T(\psi_{\theta,h}) \tag{65}$$

The best error bound we can get, by the use of the Gronwall Lemma, is the following :

$$||\psi^n_{\theta,h} - \psi_{\theta,h}(t_n)||_{1,\Omega} \leq Cexp(At_n) \quad [||\psi^0_{\theta,h} - \psi_{\theta,h}(0)|| + h^k] \tag{66}$$

Let us then suppose that the stationary solution is obtained at the time T = t_n we then have the following inequality :

$$||\psi^N_{\theta,h} - \psi_{\theta,h}||_{1,\Omega} \leq Cexp(AT) \quad [||\psi^0_{\theta,h} - \psi_{\theta,h}(0)|| + h^k] \tag{67}$$

The initial error and the interpolation error are multiplied by a factor exp (AT) which is quickly growing with T . This is an heuristic reason why the fixed point iteration algorithm gives much better results that a time dependent approach when one's is only interested by the stationary solution

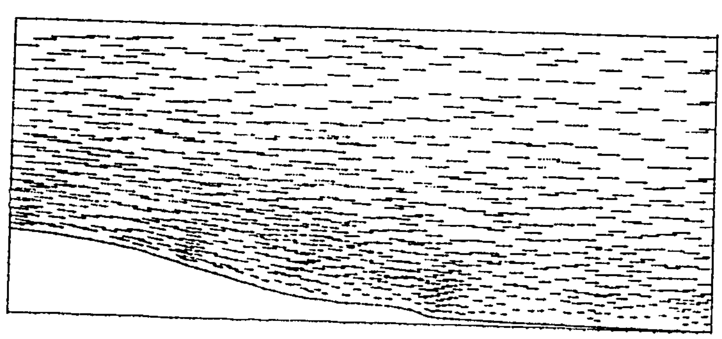

Figure 8.The velocity field of the stationary solution computed by our iterative method.

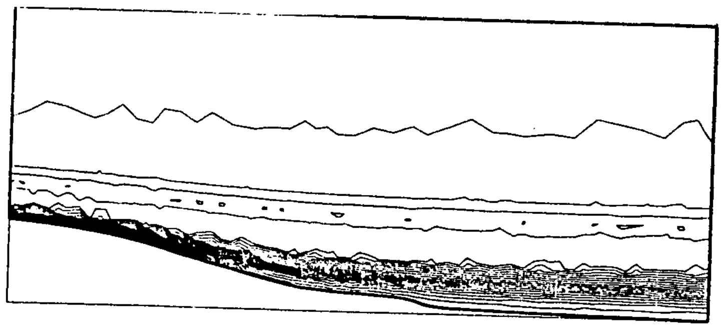

Figure 9.The vorticity ($\frac{\omega_\theta}{r}$) field of the stationary solution computed by our iterative method.

7. Modelization of a Duct Propeller .

In order to modelize the presence of stators and rotors, we introduce jumps of the angular velocity V_θ and, for the rotors only, a jump of the pressure .

Let us say a few words about the computation of the pressure in our model . Pressures are convected along the stream lines from the upstream boundary to the downstream boundary . Then, when a stream line go accross a rotor, we add a jump of pressure in order to modelize the propeller effect. That increases the velocity in the duct through the Kutta-Joukovski condition. And we also need, in that case, to introduce some viscosity effects at the trailing edge of the duct.

After many numerical experiments and computational works, we determined two practical solutions:

First, we introduce some amount of vorticity at the trailing edge in order to maintain the jump of axial velocity up and down to the trailing edge of the duct . The physically correct value for that jump was choosen as follow :

$$[\omega_\theta] = \frac{1}{\rho V_z}\frac{\partial P}{\partial r} \tag{68}$$

This solution gives good results but it may lead to computational instabilities .

The second solution prevents instabilities . We make a first computation to determine the geometry of the stream line passing by the trailing edge point . Then we completely compute the flow with this imposed stream line .

Both methods have been compared and give similar results.

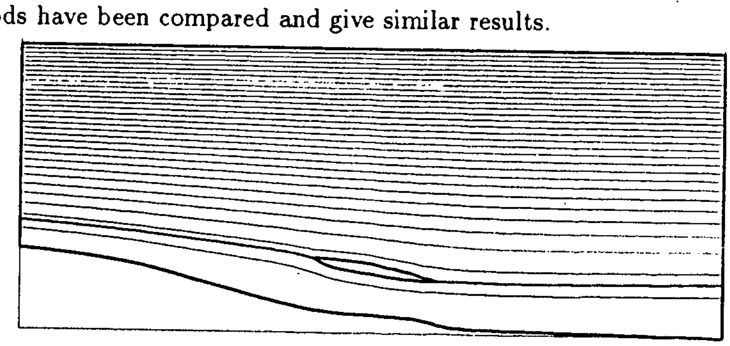

Figure 10.The stream lines of the stationary solution computed by our iterative method in the case of a duct propeller.

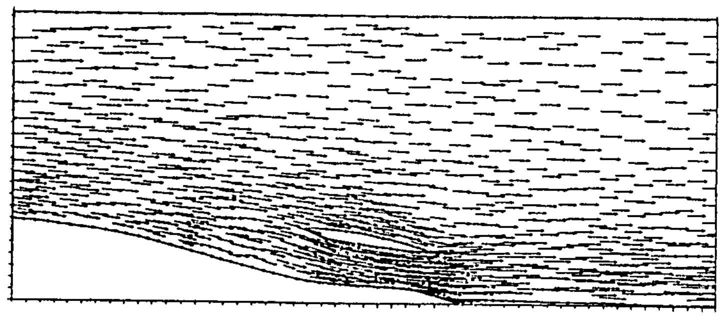

Figure 11.The velocity field of the stationary solution computed by our iterative method in the case of a duct propeller.

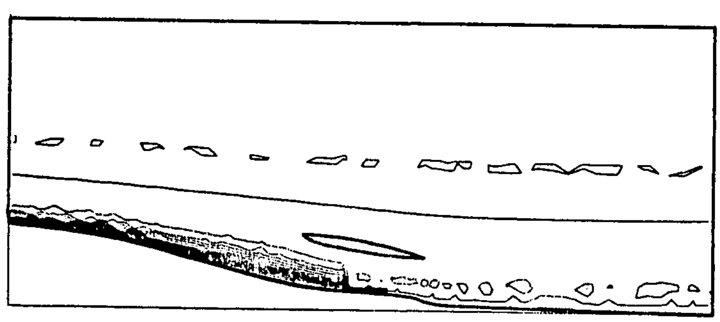

Figure 12.The vorticity ($\frac{\omega_\theta}{r}$) field of the stationary solution computed by our iterative method in the case of a duct propeller.

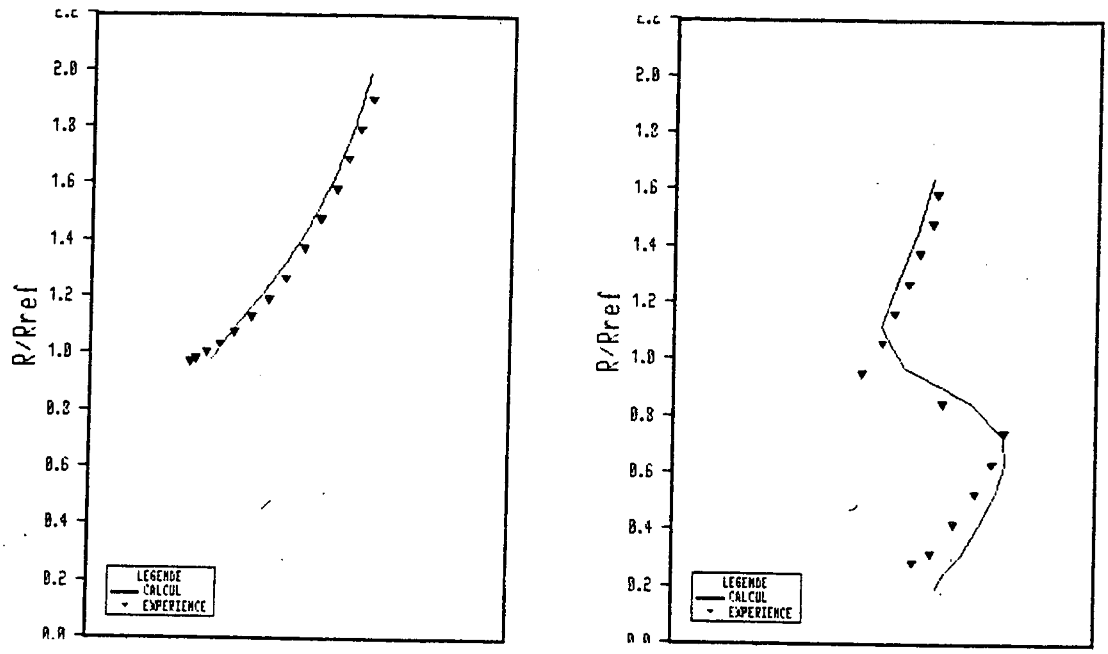

Figure 13. Comparison between computed and experimental velocity profiles before and after the propeller made by B. Goirand at the Bessin des Carenes in Paris.

More detailled discussions, complete tests and a comparison with real experiments made at the Bassin des Carenes in PARIS by B. Goirand are to appear .

Conclusion

In this paper, we presented a stable, precise, and very fast solver, based on the characteristics method, of the stationary axisymmetric Euler equation. Its stability and computing time perfomances are well adapted to "trial and error" procedures in engineering design. This scheme has been successfully used to compute an internal-external axisymmetric flow and to determine the whole propulsive performances, specially duct thrust and interaction betwen propulsor and stern. Comparisons with experiments made at the Bassin des Carenes in Paris have shown very good agreement between measures and calculus.

We are now developing a finite element blade to blade flows calculation in order to produce an automatic complete quasi 3 d solver. Our purpose is again to obtain a low time consuming, simple, stable, numerical code in order to use it in a engineering design context.

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